CONTRIBUTIONS REGARDING ANALYTICAL METHODS FOR DRIVING CURRENTS DISTRIBUTION DETERMINATION IN THREE DIMENSIONAL RECTANGULAR PLATES MEDIUMS

Daniela MINESCU

"Stefan cel Mare" University of Suceava

str.Universitatii nr. 1, RO-720225 Suceava

mind@eed.usv.ro

Abstract – *The paper presents some analytical methods for solving the driving currents distribution in rectangular plates. Considering a rectangular conducting plate with a negligible (worthless) thickness, which is connected to a power supply through two conducting terminals, the paper analyses the stationary regime in order to obtain the values of electrical potential in every point of the plate. There are considered different Neumann boundary conditions, obtained for the cases of equal or non equal conducting terminals, and the symmetric and non symmetric position of the terminals. The problem is usually solved using numerical methods, especially based on finite elements. The paper presents an exact mathematical solution using an analytical method - the separation of variables in rectangular coordinates. There is also presented Green's functions method (also known as the source function or influence function) and the results obtain using a rapidly convergent modified Green's function for Lapalce's equation. For the cases analyzed, the solution is expressed using Fourier series. Schwarz-Christoffel mapping guide to elliptic integrals, so that the solution can be performed based on the specialized toolbox of MathLAB - SC.*

Keywords - *Laplace's equation, rectangular region, modified Green's function, separation of variable.*

Introduction

The most satisfactory solution of a field problem is an exact mathematical one. Although in many practical cases such an analytical solution cannot be obtained and we must resort to numerical approximate solution, analytical solution is useful in checking solutions obtained from numerical methods. Also, one would hardly appreciate the need for numerical methods without first seeing the limitations of the classical analytical methods. The paper presents some of the most commonly used analytical methods in solving electromagnetic related problems: separation of variable, conformal mapping and Green's functions method.

The problem to solve in the next session consists in a rectangular plate witch dimensions are *a* and *b*, and the thickness *g*, as depicted in fig.1. The plate is connected to a power supply through two conducting stripes. For the beginning, we consider the same dimensions for both of the stripes and the position of them axial and symmetrical. The rectangular plate has a known conductivity, σ.

The separation of variable

Perhaps the most powerful analytical method is the separation of variable [1], [5]. Basically, it entails seeking a solution, which breaks up into a product of functions, each of which involves only one of the variables.
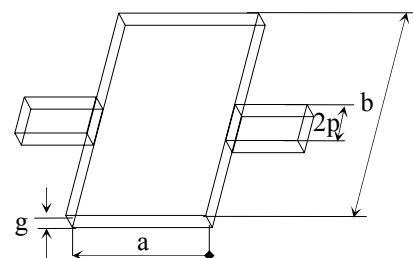

Fig.1- Rectangular plate with symmetrical stripes

The study for the case in fig.1, consists in solving Laplace's equation in two rectangular coordinates (because g<<a, b we can neglect the third dimension, and study the potentials in xOy plane. Hence, we must solve Laplace's equation inside the rectangle subject to inhomogeneous Neumann boundary conditions. Since Laplace's equation is a linear homogeneous equation, applying the superposition principle can solve the problem. If we let $V = V_1 + V_2$, we might induce the problem to two simpler problems, each of which being associated with one of the inhomogeneous conditions. Then, we must solve:

$$\frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_1}{\partial y^2} = 0 \qquad (1)$$

subject to:

$$\left.\frac{\partial V_1}{\partial y}\right|_{y=0} = 0$$

$$\left.\frac{\partial V_1}{\partial y}\right|_{y=b} = 0$$

$$\left.\frac{\partial V_1}{\partial x}\right|_{x=a} = 0 \qquad (2)$$

$$\left.\frac{\partial V_1}{\partial x}\right|_{\substack{x=0 \\ y\in\left(0,\frac{b}{2}-p\right)\cup\left(\frac{b}{2}+p,b\right)}} = 0$$

$$\left.\frac{\partial V_1}{\partial x}\right|_{\substack{x=0 \\ y\in\left(\frac{b}{2}-p,\frac{b}{2}+p\right)}} = \frac{I}{\sigma \cdot p \cdot g}$$

and the same Laplace' equation (1) subject to:

$$\left.\frac{\partial V_2}{\partial y}\right|_{y=0} = 0$$

$$\left.\frac{\partial V_2}{\partial y}\right|_{y=b} = 0$$

$$\left.\frac{\partial V_2}{\partial x}\right|_{x=0} = 0 \qquad (3)$$

$$\left.\frac{\partial V_2}{\partial x}\right|_{\substack{x=a \\ y\in\left(0,\frac{b}{2}-p\right)\cup\left(\frac{b}{2}+p,b\right)}} = 0$$

$$\left.\frac{\partial V_2}{\partial x}\right|_{\substack{x=a \\ y\in\left(\frac{b}{2}-p,\frac{b}{2}+p\right)}} = \frac{I}{\sigma \cdot p \cdot g}$$

The general solution for potential will be:

$$V_1(x,y) = \frac{I}{\sigma \cdot p \cdot g} \cdot x - \frac{2 \cdot I \cdot b}{\sigma \cdot p \cdot g} \cdot$$
$$\sum_{k=1}^{\infty} \frac{\cos\left(\frac{k\pi}{2}\right) \cdot \sin\left(\frac{k\pi p}{b}\right)}{(k\pi)^2 \cdot sh\left(\frac{k\pi a}{b}\right)} \cdot \cos\left(\frac{k\pi y}{b}\right) \cdot ch\left(\frac{k\pi(x-a)}{b}\right)$$

$$(4)$$

in the first case, and

$$V_2(x,y) = -\frac{2 \cdot I \cdot b}{\sigma \cdot p \cdot g} \cdot \sum_{k=1}^{\infty} \frac{\cos\left(\frac{k\pi}{2}\right) \cdot \sin\left(\frac{k\pi p}{b}\right)}{(k\pi)^2 \cdot sh\left(\frac{k\pi a}{b}\right)} \cdot$$
$$\cos\left(\frac{k\pi y}{b}\right) \cdot ch\left(\frac{k\pi x}{b}\right) \qquad (5$$

for the second one. *I* represent the total electric driving current through the cross section of the
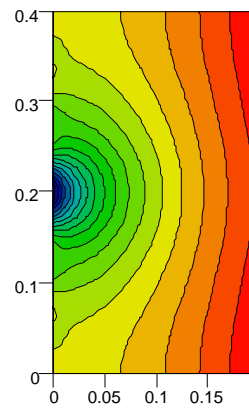

Fig.2 Equipotentials   Fig.3 Equipotentials
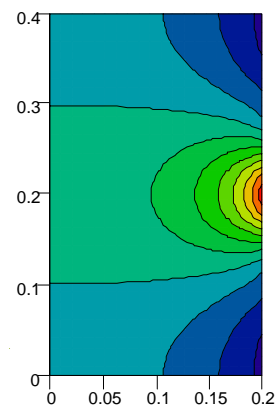calculated from (4)   calculated from (5)

stripes and σ is the electric conductivity. The image of potential distribution V_1, V_2 (using MathCad) are presented in fig.2, 3.
Adding those two solutions, based on the superposition principle, we calculate the potential in each point of the plate. These values are represented in fig.4.
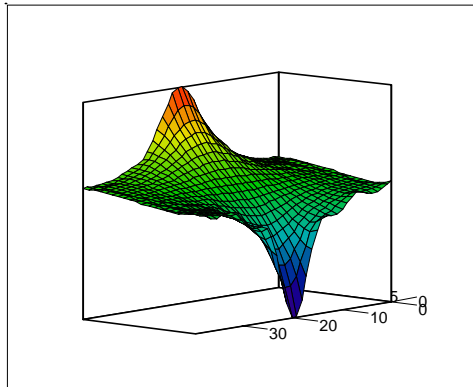
Fig.4 – Potential distribution in the rectangular plate with axial equal stripes

The calculus can be repeated for different boundary conditions. For example, if the stripes are placed with a certain eccentricity, like it is shown in fig 5, the potential obtained using the separable variable method and the superposition principle is:

$$V(x,y) = \frac{Ix}{\sigma p g} - \frac{4Ib}{\sigma g} \sum_{k=1}^{\infty} \left\{ \frac{cos\left(\frac{k\pi\pi}{b}\right)}{(k\pi)^2 sh\left(\frac{k\pi\pi}{b}\right)} \cdot \left[\frac{sin\left(\frac{k\pi(b-p_1)}{b}\right)ch\left(\frac{k\pi(x-a)}{b}\right)}{p_1} + \right.\right.$$

$$\left.\left. + \frac{sin\left(\frac{k\pi p_2}{b}\right) \cdot ch\left(\frac{k\pi x}{b}\right)}{p_1} \right] \right\} \qquad (6)$$

where p1 and p$_2$ are the dimensions of the two stripes. These values are represented in fig.6.
Using the superposition principle, it can be solved almost any field configuration of rectangular plate. In fig.7 is presented the distribution of electric potential the case in witch the stripes are equal.
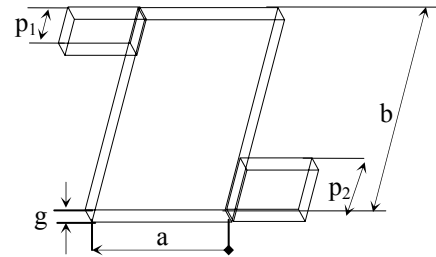

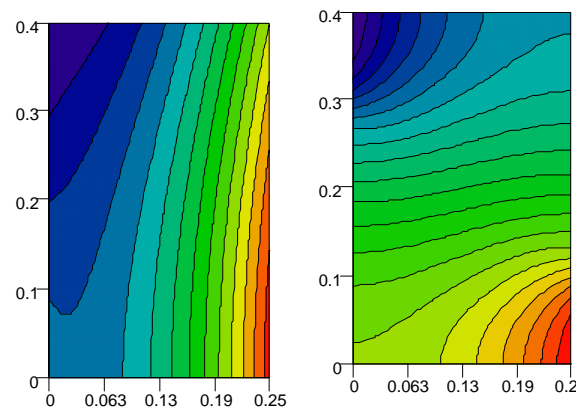Fig.5- Rectangular plate with unsymmetrical inequal stripes

Q

Fig.6. Equipotentials for unequal eccentrically eccentrically stripes

Fig.7. Equipotentials for equal stripes

Green's functions

A more systematic means of obtaining an integral equation from a partial differential equation is by constructing an auxiliary function known as the Green's function, or the source function. This is the kernel function obtained from a linear boundary value problem and forms the essential link between the differential and integral formulations.
To obtain the field caused by a distributed source by the Green's functions technique, we find the effects of each elementary portion of source and sum them up. If $G(\vec{r},\vec{r'})$ is the field at the observation point (or field point) $\vec{r}$ caused by a unit point source at the source point $\vec{r'}$, then the field at $\vec{r}$ by a source distribution $g(\vec{r'})$ is the

integral of $g(\vec{r}')\cdot G(\vec{r},\vec{r}')$ over the range of $\vec{r}'$ occupied by the source. The function G is the Green's function, witch represents the potential at $\vec{r}$ due to a unit point charge at $\vec{r}'$. For the problem in study, the points are defined in Cartesian coordinates.

The Green's Function $G=G(x,y|x',y')$ for the rectangular region D with vertices *(0,0), (a,0), (0,b)* and *(a,b)* gives the potential at *(x,y)* due to a unit current source at the point *(x',y')*, and is the solution of the differential equation

$$-\nabla^2 G = \delta(x-x',y-y') - \frac{1}{A} \qquad (7)$$

where δ is the Dirac delta function and *A* is the area of the rectangle, subject to the boundary condition that the normal derivative:

$$\frac{\partial G}{\partial n}(x,y|x',y') = 0 \qquad (8)$$

for all points (x,y) on the boundary of the rectangular region, D. The function *G* has the Fourier expansion:

$$G(x,y|x',y') = \frac{4}{ab}\cdot\sum_{m,n=0}^{\infty}\gamma_{m,n}$$

$$\frac{cos\left(\frac{m\pi x}{a}\right)cos\left(\frac{m\pi x'}{a}\right)cos\left(\frac{n\pi y}{b}\right)cos\left(\frac{n\pi y'}{b}\right)}{\left(\frac{mx}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \qquad (9)$$

where the coefficients are $\gamma_{00}=0$, $\gamma_{0n}=\gamma_{m0}=\frac{1}{2}$ and $\gamma_{mn}=1$ for $m>0$ and $n>0$. The solution of Laplace's equation in the rectangular region, with Neumann boundary condition is:

$$V(x,y) = \oint_C G(x,y|x',y')\cdot\frac{\partial V}{\partial n}\cdot dl + V_0 \qquad (10)$$

where *dl* is expressed in terms of *x'* and *y'*, *C* is the rectangle contour and V_0 is the mean value of *V* over the domain. Although in principle, this solution gives the required potential for any suitable problems, in practice, this is of very limited usefullness, owing to the extremely ppoor convergence proprieties of series expressed in (9).

It is possible to derive a much more rapidly converging series by application of known results from the theory of Fourier series and integrals, as shown in [4].

When the coefficients in (9) are assigned their appropriate values, the expresion of Green's function along an electrode embedded in the boundary between endpoints y_1 and y_2, can be written as:

$$G(x,y|x',y') = \frac{1}{a}n_y(y|y') +$$

$$+\frac{2}{\pi}\sum_{k=1}^{\infty}cos\left(\frac{k\pi x}{a}\right)cos\left(\frac{k\pi x'}{a}\right)Q_k(y,y') \qquad (11)$$

where:

$$n_y(y|y') = \begin{cases} \dfrac{1}{3}b - y' + \dfrac{y^2+y'^2}{2b}, & 0 \le y < y' \\ \dfrac{1}{3}b - y + \dfrac{y^2+y'^2}{2b}, & y' < y \le b \end{cases} \qquad (12)$$

and

$$Q_k(y,y') \equiv$$

$$\equiv \begin{cases} \dfrac{cosh\dfrac{k\pi}{a}(b-y')cosh\dfrac{k\pi y}{a}}{k\,sinh\dfrac{k\pi b}{a}}, & 0 \le y < y' \\ \dfrac{cosh\dfrac{k\pi}{a}(b-y)cosh\dfrac{k\pi y'}{a}}{k\,sinh\dfrac{k\pi b}{a}}, & y' < y \le b \end{cases} \qquad (13)$$

Using (10) for the calculus of electric potential distribution, for the case of a single electrod centred on the wall *x=0*, and assuming the current density across this to be uniform, we obtain:

$$V(x,y) = \frac{-pb^2}{12ab} + \frac{p}{ab}\left(\frac{p^2}{3} - y^2\right) + \sum_{k=1}^{\infty}\left[\frac{4a}{(k\pi)^2}\right.$$

$$\left.\frac{cos\left(\frac{k\pi x}{a}\right)cosh\left(\frac{k\pi y}{b}\right)cos\left(\frac{k\pi b}{2a}\right)sin\left(\frac{k\pi p}{a}\right)}{sinh\left(\frac{k\pi b}{a}\right)}\right] \qquad (14$$

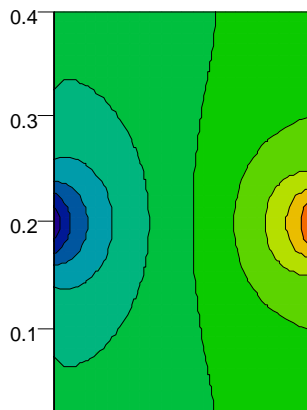The equipotential drawn based on this relation are represented in fig. 8.

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Fig.8- Equipotentials calculate with Green's function

Schwarz-Christoffel mapping

The method of conformal mapping can offer some advantages: the field can be determined correctly by using complex analytical equations. This is very suitable for a computer simulation.

The conformal mapping is a complex function, witch maps (transforms) the given configuration (in which the field should be determined) into a simple configuration (in which the field can be determined analytically), or vice versa. The simple configuration is usually an infinite half-plane.

A special kind of conformal mapping, known as Schwarz-Christoffel mapping, is suitable to analyze configurations with right angles, as it is the rectangular plate.

The Schwarz-Christoffel mapping $z = f(w)$ maps the real axis of a $w = u + j \cdot v$ coordinate system onto the boundary of the polygon in the $z = x + j \cdot y$ coordinate system, so that the upper half plane in w is mapped into the interior of the polygon in z.

The Schwarz-Christoffel mapping is given by the integral:

$$z = f(w) = S \int \prod_{k=1}^{n-1} (w - w_k)^{-\frac{\alpha_k}{\pi}} dw + C \qquad (15)$$

Here, S and C are unknown integration constants (to be found from the geometry of the configuration), n is the number of the polygon corners, w_1, …, w_n are points on the real axis of the w coordinate system, corresponding to the polygon corners z_1, …, z_n, and α_k are interior polygon angles.

If $n \le 2$, the SC integral can be solved by using the elementary complex function. If $2 < n \le 4$, the SC integral can be solved using elliptic functions. For $n > 4$, the SC integral cannot be solved analytically.

The SC integral corresponding to the rectangular region from fig.1, is:

$$z(w) = S \int_0^w \frac{dw}{\sqrt{(1 - w^2)(1 - k^2 w^2)}} + C$$

(16)

where $\pm \dfrac{1}{k}$ represent the coordinates of two of the rectangle corners in w plane. As it was already said, the calculus of this integral can be made using the elliptic function of second kind, $E(w, k)$. By the definition of these, the solution will be:

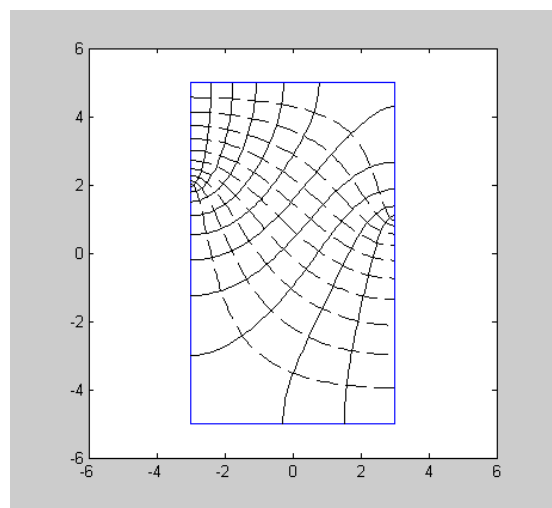$$z = f(w) = S \cdot E(w, k) + C \qquad (17)$$

Fig.9- Equipotentials and driving currents lines using SC conformal mapping from Matlab

The analytical calculus of elliptical integrals can be done only for certainness points, using mathematical tables.

There are also two sets of closed analytic functions for the approximate calculus of the complete elliptic integrals using series expand [6].

Another way to solve the problem is to use the Schwarz-Christoffel Toolbox from Matlab [2]. This is the way used to obtain the mappings in fig.9, corresponding to the same cases analysed in fig.6.

Conclusions

The paper shows a few methods suitable for the calculus of electric potential distribution in a rectangular domain with different boundary conditions.

Due to the problem geometry, these were applied three different methods. The first one, the separation of variables, is an analytical method, which calculates the solution as an infinite Fourier series expound. The potential was determined considering the first ten terms of the series.

The difficulties in solving the electrical potential distribution in the rectangular plate using the Green's functions consist of writing these functions. For the problem geometry it was expressed using the Fourier expansion.

The Schwarz-Christopher conformal mapping is based on the theory of the analytic complex functions, but, to solve the appropriate integral, the elliptic functions have to be used. Although the mapping is analytical, the final solution for the field is numerical, and it was obtained very fast and easy using the specialised Mat lab Toolbox.

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